NATIONAL BUREAU OF STANDARDS REPORT

2545

A PROPERTY OF THE NORMAL DISTRIBUTION RELATED TO
A THEOREM OF S. BERNSTEIN

by

Eugene Lukacs and Edgar P. King

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NBS PROJECT **NBS REPORT**

1103-10-1107 4 June 1953 2545

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1. Summary. The following theorem is proved.

Let $X_1, X_2, \cdots, X_n$ be n independently (but not necessarily identically) distributed random variables, and assume that the n^{th} moment of each X_i $(i = 1,2,\ldots,n)$ exists. The necessary and sufficient conditions for the existence of two statistically independent linear forms $Y_1 = \sum_{s=1}^{n} a_s X_s$ and $Y_2 = \sum_{s=1}^{n} b_s X_s$ are:

(A) Each random variable which has a nonzero coefficient in both forms is normally distributed.

(B) $\sum_{s=1}^{n} a_s b_s \sigma_s^2 = 0$.

Here σ_s^2 denotes the variance of X_s $(s = 1,2,\ldots,n)$.

For $n=2$ and $a_1=b_1=a_2=1$, $b_2=-1$ this reduces to a theorem of S. Bernstein [1] (see also [3]) which was also proved by M. Kac [4] in measure theoretic terms. Another particular case of the theorem is stated without proof in a recent paper by Yu. V. Linnik [5].

2. Introduction. We consider two linear forms

$$(1) \qquad Y_1 = \sum_{s=1}^{n} a_s X_s \; ; \quad Y_2 = \sum_{s=1}^{n} b_s X_s$$

in the n independently distributed random variables $X_1, X_2, \cdots, X_n$.

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We arrange the variables so that the first p $(X_1, X_2, \cdots, X_p)$ have **nonzero** coefficients in both forms and the remaining $(n-p)$ have zero coefficients in one form or the other. Clearly $0 \leq p \leq n$. When $p=0$, Y_1 and Y_2 are trivially independent; when $p=1$, Y_1 and Y_2 cannot be independent. For $p \geq 2$, it is clear that the statistical independence of the original linear forms (1) is completely equivalent to the independence of the forms $Z_1 = \sum_{s=1}^{p} a_s X_s$ and $Z_2 = \sum_{s=1}^{p} b_s X_s$. This means that when $p < n$ the distributions of the random variables $X_{p+1}, \cdots, X_n$ do not affect the independence of Y_1 and Y_2. This is why the theorem contains only a statement about the distributions of those random variables with **nonzero** coefficients in both forms.

If for some pairs of corresponding coefficients, say the first r $(1<r<p)$, the relation

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_r}{b_r} = C \tag{2}$$

holds, then we can rewrite Z_1 and Z_2 as

$$Z_1 = C(b_1X_1 + \cdots + b_rX_r) + a_{r+1}X_{r+1} + \cdots + a_pX_p \text{ , and}$$
$$Z_2 = b_1X_1 + \cdots + b_rX_r + b_{r+1}X_{r+1} + \cdots + b_pX_p \text{ .}$$

Introducing the new variable $\widetilde{X}_1 = b_1X_1 + \cdots + b_rX_r$, we see that the independence of Y_1 and Y_2 is equivalent to the independence of the forms $\widetilde{Z}_1 = C\widetilde{X}_1 + a_{r+1}X_{r+1} + \cdots + a_pX_p$ and $\widetilde{Z}_2 = \widetilde{X}_1 + b_{r+1}X_{r+1} + \cdots + b_pX_p$. If the theorem holds for the forms $\widetilde{Z}_1$ and $\widetilde{Z}_2$, Cramér's theorem [2] shows that the normality of $\widetilde{X}_1$ implies the normality of the random variables $X_1, X_2, \cdots, X_r$. We proceed in the same manner if there are several groups of random variables for which a relation

of type (2) holds. Hence our problem reduces to the study of the independence of two linear forms whose coefficient matrix contains no vanishing minor or order 2.

Finally it is clear that the independence of Y_1 and Y_2 is equivalent to the independence of the forms $\tilde{Y}_1 = \sum_{s=1}^{n} a_s(X_s - E[X_s])$ and $\tilde{Y}_2 = \sum_{s=1}^{n} b_s(X_s - E[X_s])$. Therefore we shall assume without loss of generality that the following conditions are satisfied:

(i) $a_s b_s \neq 0 \quad (s = 1,2,\ldots,n)$

(ii) $a_s b_t - a_t b_s \neq 0$ for all $s \neq t \quad (s,t = 1,2,\ldots,n)$

(iii) $E[X_s] = 0 \quad (s = 1,2,\ldots,n)$.

3. <u>The functional equation for the characteristic functions.</u>

Denote the distribution function of the random variable X_s $(s=1,\ldots,n)$ by $F_s(x)$ and the corresponding characteristic function by $f_s(t)$. Let $h(u,v)$ be the c.f. of the joint distribution of Y_1 and Y_2 and write $h_1(u) = h(u,0)$, $h_2(v) = h(0,v)$. Clearly $h_1(u)$ and $h_2(v)$ are the c.f.'s of the distributions of Y_1 and Y_2, respectively.

We prove first that our conditions are necessary; that is, we assume that Y_1 and Y_2 are statistically independent. In terms of characteristic functions this means

$$h(u,v) = h_1(u)h_2(v) \quad . \tag{3}$$

Further, because $X_1,\ldots,X_n$ are independent, we have

$$h_1(u) = \prod_{s=1}^{n} f_s(a_s u) \quad , \tag{4}$$

$$h_2(v) = \prod_{s=1}^{n} f_s(b_s v) \quad , \tag{5}$$

$$h(u,v) = \prod_{s=1}^{n} f_s(a_s u + b_s v) \quad . \tag{6}$$

Finally, substituting (4), (5), and (6) in (3) we obtain the following functional equation in the characteristic functions:

$$(7) \qquad \prod_{s=1}^{n} f_s(a_s u+b_s v) = \prod_{s=1}^{n} f_s(a_s u) f_s(b_s v) \quad .$$

4. <u>The differential equations for the cumulant generating functions</u>. The general procedure for determining the explicit form of the characteristic functions $f_s(t)$ will be to differentiate the logarithm of (7) r times $(r = 1,2,\ldots,n)$ with respect to u, set u=0, and solve the resulting n differential equations for $\ln f_s(t)$ $(s = 1,\ldots,n)$.

We first note that $f_s(o) = 1$ $(s = 1,\ldots,n)$ and that $f_s(t)$ is a continuous function of t. Therefore there exists a neighborhood of the origin in which all the factors occuring in (7) are different from zero. This neighborhood could of course be the entire plane. In the following derivation we restrict the values of u and v to this neighborhood; then we may take the logarithm of both sides of (7) and obtain

$$(9) \qquad \sum_{s=1}^{n} \phi_s(a_s u+b_s v) = \sum_{s=1}^{n} \phi_s(a_s u) + \sum_{s=1}^{n} \phi_s(b_s v) \; ,$$

$$\text{where } \phi_s(x) = \ln f_s(x) \; .$$

Differentiating (9) r times with respect to u and setting u=0 yields

$$(10) \qquad \sum_{s=1}^{n} \left[\frac{\partial^r}{\partial u^r} \phi_s(a_s u+b_s v) \right]_{u=0} = \sum_{s=1}^{n} \left[\frac{d^r}{du^r} \phi_s(a_s u) \right]_{u=0} \; .$$

Letting $z_s = a_s u$, we find that the typical term on the left side of (10) becomes

$$(11)\qquad \left[\frac{\partial^r}{\partial u^r}\,\phi_s(a_s u+b_s v)\right]_{u=o} = a_s^r\left[\frac{\partial^r}{\partial z_s^r}\,\phi_s(z_s+b_s v)\right]_{z_s=0} .$$

Employing the substitution $\psi_s(v) = \phi(b_s v)$, (11) becomes

$$(12)\qquad \left[\frac{\partial^r}{\partial u^r}\,\phi_s(a_s u+b_s v)\right]_{u=0} = \left(\frac{a_s}{b_s}\right)^r \frac{d^r}{dv^r}\,\psi_s(v) .$$

Similarly the typical term on the right side of (10) becomes

$$(13)\qquad \left[\frac{d^r}{du^r}\,\phi_s(a_s u)\right]_{u=0} = a_s^r\left[\frac{d^r}{dz^r}\,\phi_s(z_s)\right]_{z_s=0} = (ia_s)^r\kappa_r^{(s)}$$

where $\kappa_r^{(s)}$ is the r^{th} order cumulant of X_s. Substituting (12) and (13) in (10) we obtain

$$(14)\qquad \sum_{s=1}^{n} \xi_s^r\,\frac{d^r}{dv^r}\,\psi_s(v) = \sum_{s=1}^{n}(ia_s)^r\kappa_r^{(s)} \qquad (r = 1,2,\ldots,n)$$

$$\text{where } \xi_s = \frac{a_s}{b_s} .$$

Differentiating (14) (n-r) times yields the system of differential equations

$$(15)\qquad \begin{cases} \displaystyle\sum_{s=1}^{n} \xi_s^r\,\frac{d^n}{dv^n}\,\psi_s(v) = 0 & (r = 1,2,\ldots,n-1) \\ \displaystyle\sum_{s=1}^{n} \xi_s^n\,\frac{d^n}{dv^n}\,\psi_s(v) = \sum_{s=1}^{n}(ia_s)^n\kappa_n^{(s)} & . \end{cases}$$

We have to determine all the distribution functions whose characteristic functions satisfy this system of differential equations and the initial conditions

$$(15a)\qquad \begin{cases} \left[\dfrac{d^r}{dv^r}\,\psi_s(v)\right]_{v=0} = (ib_s)^r\kappa_r^{(s)} & r = 1,2,\ldots,n-1 \\ \psi_s(0) = 1 . \end{cases}$$

We now define

$$D_n = \begin{vmatrix} \xi_1 & \cdots\cdots & \xi_n \\ \xi_1^2 & \cdots\cdots & \xi_n^2 \\ \cdots & \cdots\cdots & \cdots \\ \xi_1^n & \cdots\cdots & \xi_n^n \end{vmatrix}$$

and denote by $D_{s,n}$ the cofactor of the element in the s^{th} column and the n^{th} row of D_n. Considering (15) as a system of n linear equations in the quantities $\frac{d^n}{dv^n} \psi_s(v)$, we obtain the solutions

(16) $$\frac{d^n}{dv^n} \psi_s(v) = \frac{D_{s,n}}{D_n} \sum_{s=1}^{n} (ia_s)^n \kappa_n^{(s)} = i^n C_{s,n} \text{ , say .}$$

Integrating (16) n times and employing the initial conditions (15a) yields

$$\psi_s(v) = \sum_{j=1}^{n-1} \frac{(ib_s)^j}{j!} \kappa_j^{(s)} v^j + \frac{C_{s,n}}{n!} (iv)^n \quad .$$

Since $f_s(b_s v) = \exp[\phi_s(b_s v)] = \exp[\psi_s(v)]$ we have

(17) $$f_s(b_s v) = \exp\left[\sum_{j=1}^{n-1} \frac{\kappa_j^{(s)}}{j!} (ib_s v)^j + \frac{C_{s,n}}{b_s^n n!} (ib_s v)^n \right] .$$

In case any of the functions $f_s(t)$ become zero for some real t, this solution is valid only in a certain neighborhood of the origin. We next show by an indirect proof that none of the functions $f_s(t)$ $(s = 1,\ldots,n)$ has a real zero; from this we can conclude that (17) is valid for all real t.

Let us therefore assume that one or more of the c.f.'s $f_s(t)$ have zeros. In this case at least one of the functions $f_s(b_s v)$ will have a zero. Denote by v_r^o the zero closest to the

origin and by $f_r(t)$ a function for which $f_r(b_r v_r^o) = 0$. For $|v| < |v_r^o|$ we have $f_s(b_s v) \neq 0$ $(s = 1,\dots,n)$ and formula (17) is valid. Let v be a real number such that $|v| < |v_r^o|$; then we have by (17)

$$f_r(b_r v) = \exp\left[\sum_{j=1}^{n-1} \frac{\kappa_j^{(n)}}{j!}(ib_r v)^j + \frac{C_{r,n}}{b_r^n n!}(ib_r v)^n\right] . \tag{18}$$

But $f_r(t)$ is a continuous function. Hence $\lim_{v \to v_r^o} f_r(b_r v) = f_r(b_r v_r^o) = 0$ by assumption. However, from (18) it is clear that

$$\lim_{v \to v_r^o} f_r(b_r v) = \exp\left[\sum_{j=1}^{n-1} \frac{\kappa_j^{(n)}}{j!}(ib_r v_r^o)^j + \frac{C_{r,n}}{b_r^n n!}(ib_r v_r^o)^n\right]$$

which is always different from zero. This is a contradiction, and hence formula (17) is valid for all values of v. Writing $t = b_s v$ we finally obtain

$$f_s(t) = \exp\left[\sum_{j=1}^{n-1} \frac{\kappa_j^{(s)}}{j!}(it)^j + \frac{C_{s,n}}{b_s^n n!}(it)^n\right] . \tag{19}$$

5. Proof of the theorem. We have determined all the solutions of the system (15) satisfying the initial conditions (15a). In order to find the distribution functions whose characteristic functions satisfy this system we must select those functions (19) which are characteristic functions. This is easily done by means of the following result due to Marcinkiewicz [6].

Theorem of Marcinkiewicz.

No function of the form $e^{a_o+a_1 z+\dots+a_r z^r}$ $(r>2)$ can be a characteristic function.

Hence the degree of the polynomial in (19) cannot exceed 2. In

case n>2 we must have

$$\kappa_j^{(s)} = 0 \qquad j = 3,4,\ldots,n-1;\ s = 1,2,\ldots,n \quad (n>3)$$

$$C_{s,n} = 0 \qquad (n>2);\quad s = 1,2,\ldots,n \ .$$

Because the factor $\frac{D_{s,n}}{D_n}$ is $C_{s,n}$ cannot vanish, these relations reduce to

$$(20) \qquad \kappa_j^{(s)} = 0 \qquad j = 3,\ldots,n-1 \qquad n>3$$

$$\sum_{s=1}^{n} a_s^n \kappa_n^{(s)} = 0 \qquad n>2 \ .$$

There is no restriction if n = 2. In view of (iii) $\kappa_1^{(s)} = 0$ also, and (19) becomes

$$(21) \qquad f_s(t) = \exp\left[-\tfrac{1}{2}\sigma_s^2 t^2\right] \qquad \text{for } n > 2 \ .$$

This shows that each X_s $(s = 1,\ldots,n)$ must be normally distributed, which is condition (A) of the theorem. All cumulants of order r>2 vanish for a normal distribution, hence equations (20) impose no additional restrictions. In case n = 2 we have

$$(22) \qquad f_s(t) = \exp\left[e^{-k/2}\, t^2\right] \qquad \text{for} \quad n = 2 \ ,$$

where k is determined from (16) and (19). The independence of Y_1 and Y_2 implies that they are uncorrelated which yields condition (b) and completes the first part of the proof.

It is easy to establish that conditions (A) and (B) are also sufficient. Assuming that (A) and (B) hold, it follows that Y_1 and Y_2 are uncorrelated and normally distributed. Hence Y_1 and Y_2 must be independent.

For $n = 2$ and $a_1 = a_2 = b_1 = 1$, $b_2 = -1$ we obtain from (22)

$$f_s(t) = \exp\left[-\left(\frac{\sigma_1^2 + \sigma_2^2}{2}\right) t^2\right] \qquad s = 1,2 \quad .$$

This shows that $\sigma_1^2 = \sigma_2^2$ and establishes Bernstein's theorem.

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